Complex Network Topologies and Synchronization

INTRODUCTION

Real networks of interacting dynamical systems -- be they neurons, power stations or lasers -- are complex. Many real-world networks are small-world [1] and/or scale-free networks [2]. The presence of a power-law connectivity distribution, for example, makes the Internet a scale-free network. The research on complex networks has been focused so far on their topological structure [3]. However, most networks offer support for various dynamical processes. In this paper we propose to study one aspect of dynamical processes in non-trivial complex network topologies, namely their synchronization behaviors.

The general question of network synchronizability, for many aspects, is still an open and outstanding research problem [4, 5]. In this context, an important contribution has been given by Pecora and Carroll in [6], where, for a network of coupled chaotic oscillators, they derived the so-called Master Stability Equation (MSE), and introduced the corresponding Master Stability Function (MSF). Consequently, the stability analysis of the synchronous manifold [6] for the network under consideration can be decomposed in two sub-problems. The first sub-problem consists of deriving the MSF for the network nodes, i.e. to study in which region of the complex plane the MSE admits a negative largest Lyapunov exponent (LE). The second sub-problem is to verify whether the eigenvalues of the so-called connectivity matrix [7] of the network, apart from the zero-eigenvalue, lie in the synchronization region(s) (see also [6, 7, 8]). This approach is particularly relevant because the MSE depends only on the nodes local dynamics and on the coupling matrix [7].

In this work, at first, we study the synchronization regions, using the properties of the MSF. Namely, in Section II it is shown that for typical systems only three main scenarios may arise as a function of coupling strength. Then, we study synchronization in complex networks topologies. Section III is devoted to the analysis of synchronization properties of networks whose topology is described by classical random networks. In Section IV we study synchronization properties of power-law random graph models. We close our paper with conclusions (Section V).

II. SYNCHRONIZATION REGIONS

Let us consider a network with N identical nodes, each being a (chaotic) oscillator. Let \( x_i \) be the m-dimensional vector of dynamical variables for the i-th node. Let the dynamics of each node be described by:

\[
\dot{x}_i = f(x_i) + \sigma \sum_{k=1}^{N} D_{ik} x_k \quad i = 1, \ldots, N
\]

where \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) describes the oscillator equations, which we assume to admit a chaotic attractor, \( \sigma \) is the overall strength of coupling, while \( D_{ik} \) are \( m \times m \) real matrices. Assume that each matrix, \( D_{ik} \), has the form: \( D_{ik} = \lambda_{ij} H \), where \( \lambda_{ij} \) is a real number defined in the following and \( H \) is a \( m \times m \) diagonal matrix, same for all nodes, called coupling matrix. The coupling matrix \( H = (h_{ij}) \) contains the information about which variables are utilized in the coupling and is defined as \( h_{ii} = 1 \), if the i-th component is coupled, and \( h_{ij} = 0 \), otherwise. Let \( x = (x_1, \ldots, x_N)^T \), \( f(x) = (f(x_1), \ldots, f(x_N))^T \). Furthermore, let the \( N \times N \) matrix \( L = (l_{ij}) \) be the Laplacian matrix, representing the connection topology of the network: \( l_{ij} = l_{ji} = 1 \) if nodes \( i \) and \( j \) are connected, \( l_{ii} = k_i \) if node \( i \) is connected to \( k_i \) other nodes, and \( l_{ij} = l_{ji} = 0 \) otherwise.

Then, we can rewrite Eq. (1) in a more compact form using the direct product of matrices:

\[
\dot{x} = F(x) + \sigma (L \otimes H) x,
\]

where \( F(x) : \mathbb{R}^{mN} \rightarrow \mathbb{R}^{mN} \) is defined as \( F(x) = (f(x_1), \ldots, f(x_N))^T \).

The matrix \( L \), which will be our main concern, is positive semi-definite and symmetric. Its smallest eigenvalue is \( \gamma_1 = 0 \). Denote by \( \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_N \) the eigenvalues of \( L \). In particular, \( \gamma_N \), is the maximal eigenvalue of \( L \).
Since $L$ is symmetric, the master stability function, in this case, has the form [6]
\[
\zeta = |J_f + \alpha H| \zeta,
\]
where $\alpha \in \mathbb{R}$ and $J_f$ is the Jacobian matrix of $f(x)$. Therefore, in this case the corresponding largest Lyapunov exponent or MSF, $\Lambda(\alpha)$, depends only on one parameter, $\alpha$. Master stability function determines the linear stability of the synchronized state; in particular, the synchronized state is stable if all eigenvalues of the matrix $L$ are in the region $\Lambda(\alpha) < 0$. We denote by $S \subseteq R$ the region where the MSF is negative and call it synchronization region. Discussions in [5] show in fact that for the system (2), the synchronization region $S$ may have one of the following forms:

1. $S_1 = \emptyset$
2. $S_2 = (\alpha_m, +\infty)$
3. $S_3 = \bigcup_{j=1}^M (\alpha_m(j), \alpha_M(j))$

Examples of these scenarios are given in [9], [10]. In the majority of cases $\alpha_m$, $\alpha_m(j)$, and $\alpha_M(j)$ turn out to be positive and, furthermore, in the case $S_1$ there is only one parameter interval $(\alpha_m(j), \alpha_M(j))$ on which $\Lambda(\alpha) < 0$. For this reason, we will limit ourselves to consider only such cases, focusing, in the remaining of this paper, on the scenarios $S_2 = (\alpha_m, +\infty)$ and $S_3 = (\alpha_m(j), \alpha_M(j))$. It is easy to see that for $S_2$ the condition of stable synchronous state is $\sigma_{\gamma_2} > \alpha_m$. For $S_3$, one can easily show that there is a value of the coupling strength $\sigma$ for which the synchronization state is linearly stable, if and only if $\gamma_N / \gamma_2 < \alpha_M / \alpha_m$. Therefore, for a large class of (chaotic) oscillators there exist two classes of networks:

1. Class-A networks: networks whose synchronization region is of type $S_2$, for which the condition of stable synchronous state is $\sigma_{\gamma_2} > \alpha$;
2. Class-B networks: networks whose synchronization region is of type $S_3$, for which this condition reads $\gamma_N / \gamma_2 < b$;

where $\alpha = \alpha_m$ and $b = \alpha_M / \alpha_m$ are constants that depend on $f$, the synchronous state $x_1 = x_2 = \ldots = x_N$ and the matrix $H$, but not on the Laplacian matrix $L$. For typical oscillators $b > 1$.

III. SYNCHRONIZATION IN CLASSICAL RANDOM NETWORKS

The primary model for the classical random graphs is the Erdős-Rényi model $G(N,q)$ [11], in which each edge is independently chosen with the probability $q$ for some given $q > 0$. Let $G(N,q)$ be a random graph on $N$ vertices.

For the model of a random graph we take a sequence of probability spaces $(\Gamma(N,q))_N$, where $q$ is a real number between 0 and 1, and $N$ is an integer. We shall assume that $q$ is fixed, but in general it may depend on $N$. The probability space $\Gamma(N,q)$ consists of all labelled simple graphs on $N$ vertices, and an edge between an arbitrary pair of vertices appears with probability $q$, i.e. $\Gamma(N,q)$ has $2^M$ elements, where $M = N(N-1)/2$, and each graph in $\Gamma(N,q)$ with $m$ edges has the probability equal to $q^m(1-q)^{M-m}$. By $P_{N,q}(X)$ we will denote the probability of an event $X \subseteq \Gamma(N,q)$ in the probability space $\Gamma(N,q)$. Let $p(G)$ mean that the graph $G$ has the property $\rho$. We say that the property $\rho$ happens asymptotically almost surely (a.a.s.), if
\[
\lim_{N \to \infty} P_{N,q}(G \in \Gamma(N,q) : p(G)) = 1.
\]

Theorem 3.1: Let $G(N,q)$ be a random graph on $N$ vertices. Then the class-A network $G(N,q)$ asymptotically almost surely synchronize for arbitrary small coupling $\sigma$ and the class-B network $G(N,q)$ asymptotically almost surely synchronize for $b > 1$.

Proof: The proof of the theorem follows from the following result [12]. Let $q$ be a fixed real number between 0 and 1. For almost every graph and every $\varepsilon > 0$
\[
\left\{ \begin{array}{l}
\gamma_2(G) > qN - \sqrt{(2+\varepsilon)pqN \log N} \\
\gamma_2(G) < qN - \sqrt{(2-\varepsilon)pqN \log N}
\end{array} \right.
\]
and
\[
\left\{ \begin{array}{l}
\gamma(N) > qN + \sqrt{(2+\varepsilon)pqN \log N} \\
\gamma(N) < qN + \sqrt{(2-\varepsilon)pqN \log N}
\end{array} \right.
\]
Therefore, for large $N$, $\gamma_2 \approx N$, while $\gamma(N)/\gamma_2$ approaches 1. Now, for class-A networks the condition for synchronization reads $\sigma > \alpha / N$ and $\sigma$ can be chosen arbitrary small. For class-B networks with $b > 1$, since $\gamma(N)/\gamma_2$ approaches 1, when $N \to \infty$, it follows that the network almost surely synchronizes.

IV. SYNCHRONIZATION IN POWER-LAW NETWORKS

We consider a random model introduced recently by Chung and Lu [13], which produces graphs with a given expected degree sequence. Therefore, this model does not produce the graph with exact given degree sequence. Instead, it yields a random graph with given expected degree sequence.

We consider the following class of random graphs with a given expected degree sequence $w = (w_1, w_2, \ldots, w_N)$. The vertex $v_i$ is assigned vertex weight $w_i$. The edges are chosen independently and randomly according to the vertex weights as follows. The probability $p_{ij}$ that there is an edge between $v_i$ and $v_j$ is proportional to the product $w_i w_j$, where $i$ and $j$ are not required to be distinct. There are possible loops at $v_i$ with probability proportional to $w_i^2$, i.e.,
\[
p_{ii} = \frac{w_i^2}{\sum_k w_k},
\]
and we assume $\sum_i w_i^2 < \sum_k w_k$. This assumption ensures that $p_{ij} \leq 1$ for all $i$ and $j$. We denote a random graph with a given expected degree sequence $w$ by $G(w)$. For example, a typical random graph $G(N,q)$ (see the previous section) on $N$ vertices and edge density $q$ is just a random graph with expected degree sequence $(q N, q N, \ldots, q N)$. The random graph $G(w)$ is different from the random graphs with an exact degree sequence such as the configuration model. We will use $d_i$ to denote the actual degree of $v_i$ in a random graph $G$ in $G(w)$, where the weight $w_i$ denotes the expected degree. The following proposition is proved in [13].
Proposition 4.1: With probability $1 - 2/N$, all vertices $v_i$ satisfy
\[
2w_i \log N \leq d_i - w_i \leq \frac{3}{2} \log N + \sqrt{\left(\frac{3}{2} \log N\right)^2 + 4w_i \log N}.
\]  
(8)

We consider the model $M(N, \beta, d, m)$, where $N$ is the number of vertices, $\beta > 2$ is the power of the power law, $d$ is the expected average degree, defined as $d = \sum w_i/N$, and $m$ is the expected maximum degree (or an upper bound for the range of degrees that obey the power law), such that $m^2 = o(Nd)$. We assume that the $i$-th vertex $v_i$ has expected degree $w_i = c(i + i_0 - 1)^{-\beta/2}$, for $1 \leq i \leq N$. Here $c$ depends on the average degree $d$ and $i_0$ depends on the maximum expected degree $m$ (see [5] for passages):
\[
c = \frac{\beta - 2}{\beta - 1} dN^{1 \beta/2}
\]  
(9)
\[
i_0 = N \left[ \frac{d (\beta - 2)}{m (\beta - 1)} \right]^{\beta - 1}.
\]  
(10)

It is easy to compute that the number of vertices of expected degree between $k$ and $k + 1$ is of order $c^{k^\beta}$, where $c' = c^{\beta - 1}(\beta - 1)$, as required by the power law.

Let $k$ be the expected minimum degree. Then
\[
k = \frac{\beta - 2}{\beta - 1} d \left[ 1 + \frac{d (\beta - 2)}{m (\beta - 1)} \right]^{-\beta - 1}.
\]  
(11)

For the considered model $d$ can be in any range greater than 1: it does not have to grow with $N$ [13]. We first consider the case when $d$ grows with $N$.

Theorem 4.2: Let $M(N, \beta, d, m)$ be a random power-law graph on $N$ vertices, for which $d$ grows with $N$ and $d/m \to 0$ when $N \to \infty$. Then class-A network $M(N, \beta, d, m)$ asymptotically almost surely synchronizes for arbitrary small coupling $\sigma$ and class-B network $M(N, \beta, d, m)$ asymptotically almost surely does not synchronize.

Proof: From [5], the following inequalities hold
\[
\frac{N}{N - 1} \Delta(M) \leq \gamma_N(M) \leq 2 \Delta(M),
\]  
(12)
where $\Delta(\cdot)$ denotes the maximum degree of a graph. It follows that for large $N$ we have $\Delta < \gamma_N \leq 2 \Delta$. Therefore, from (8) we have $\gamma_N(M) \approx \Delta \approx m$ for large $N$ [5]. Equation (11) can be rewritten as
\[
k \approx d \left[ 1 + \frac{d (\beta - 2)}{m (\beta - 1)} \right]^{-\beta - 1}.
\]  
(11)

Since $d \ll m$, we have $k \approx d$. Therefore, when $d$ grows with $N$, the minimum expected degree $k$ also grows with $N$.

It is proven in [14] that the function $\gamma_2(G)$ is non-decreasing for graphs with the same set of vertices, i.e. $\gamma_2(G_1) \leq \gamma_2(G_2)$ if $G_1 \subseteq G_2$ and $G_1, G_2$ have the same set of vertices. Let $G_2$ be our $M(N, \beta, d, m)$ random graph and $G_1$ be a $k$-regular random graph which has the same set of vertices as $G_2$. Then $\gamma_2(M) \geq \gamma_2(G_1)$.

According to [15] and [16] (see also [17]), we have
\[
\gamma_2(M) \geq \gamma_2(G_1) \geq k - \sqrt{\frac{3}{4} k^2 - d (\ln 2 - \sqrt{k \ln 2})}.
\]  
(13)

On the other hand, from the following inequality (see [5])
\[
\gamma_2 \leq \frac{N}{N - 1} \delta
\]  
(14)
and (8), it follows that for large $N$,
\[
\gamma_2(M) \leq \frac{N}{N - 1} \delta \approx \delta \approx k,
\]  
(15)
where $\delta$ is the minimum degree of the graph. Combining (13) and (15) we find that $\gamma_2(M)$ can be approximated with $k$.

If $d$ grows with $N$, since $\gamma_2$ also grows with $N$ we conclude that the class-A network $M(N, \beta, d, m)$ asymptotically almost surely synchronize for arbitrary small coupling $\sigma$. Since $b$ is a finite number, from $\gamma_N/\gamma_2 \approx m/k$, we see that for sufficiently large $N$, almost every class-B network $M(N, \beta, d, m)$ does not synchronize.

Fig. 1. $\gamma_2$ versus $N$ for the model $M(N, \beta, d, m)$ with $\beta = 3, d = 7$, and $m = 30$.

Now we consider the case $d \ll \infty$. Since, in this case, we could not obtain analytical bounds for $\gamma_2$ and $\gamma_N$ we provide numerical examples. Consider the model $M(N, \beta, d, m)$ with $\beta = 3, d = 7$, and $m = 30$. Figures 1 to 3 show the $\gamma_2$, $\gamma_N$, and $\gamma_N/\gamma_2$ versus $N$. The figures are obtained by simulating graphs composed of 1000 to 1200 nodes, with a step of 10 nodes. For each case, 10 different simulations are computed and the mean value is presented as a dot (solid line is a curve fitting the dots). Note that the actual maximum degree $\Delta$ may differ from the expected maximum degree $m$. Consider now a class-A network with $\alpha = 1$ and a class-B network with $b = 40$. From Fig. 1 one can compute the value of $\gamma_2$ for $N = 1200$, $\gamma_2 = 0.31$, and therefore, the network synchronizes for $\sigma > 3.23$. Moreover, from Fig. 3 one can compute the value of $\gamma_N/\gamma_2$ for $N = 1200$, which is approximately
Fig. 2. \(\gamma_N / \gamma_2\) versus \(N\) for the model \(M(N, \beta, d, m)\) with \(\beta = 3, d = 7, \text{ and } m = 30\).

![Graph](image)

\[\gamma_N / \gamma_2 = 107.\] Consequently, since \(b < 107\), the class-B network does not synchronize.

Let us write \(\sigma_c = a / \gamma_2\) and \(b_c = \gamma_N / \gamma_2\). \(\sigma_c\) and \(b_c\) are critical values for which the network may synchronize, in other words, if \(\sigma > \sigma_c, b > b_c\), then the class-A (class-B) network synchronizes. The proof of Theorem 4.2 suggests that the critical values may be approximated as \(\sigma_c \approx a / k\) and \(b_c = m / k\) provided that \(k\) and \(d\) are close to each other. For example, consider a network composed by \(N = 1200\) nodes with \(d = 20, m = 200, \beta = 3\), for which \(k \approx 9.99\). Then we have (for \(a = 1\) and \(b = 40\)) \(\sigma_c \approx 0.10\) and \(b_c \approx 20.02\). Simulating such a network, the following actual eigenvalues have been obtained: \(\gamma_2 (a \sigma_t) \approx 7.61, \gamma_N (a \sigma_t) \approx 196.43, \text{ and } (\gamma_N / \gamma_2) (a \sigma_t) \approx 25.83\). It follows that the actual critical values are \(\sigma_c (a \sigma_t) \approx 0.13\) and \(b_c (a \sigma_t) \approx 25.83\). In this case, since \(b = 40\), both class-A and class-B networks synchronize.

V. CONCLUSION

In this paper we studied synchronization in networks with different topologies.

We showed that for a large class of oscillators there exist two classes of networks: class-A: networks for which the condition of stable synchronous state is \(\sigma \gamma_2 > a\), and class-B: networks for which this condition reads \(\gamma_N / \gamma_2 < b\), where \(a\) and \(b\) are constant that depend on local dynamics, synchronous state and the coupling matrix, but not on the Laplacian matrix of the graph describing the topology of the network.

Let \(G(N, q)\) be a classic random graph (Erdős-Rényi model) on \(N\) vertices. We proved that for sufficiently large \(N\), the class-A network \(G(N, q)\) almost surely synchronize for arbitrary small coupling \(\sigma\). For sufficiently large \(N\), almost every class-B network \(G(N, q)\) with \(b > 1\) synchronizes.

Let \(M(N, \beta, d, m)\) be a random power-law graph on \(N\) vertices. We proved that for sufficiently large \(N\), the class-A network \(M(N, \beta, d, m)\) almost surely synchronize for arbitrary small coupling \(\sigma\). For sufficiently large \(N\), almost every class-B network \(M(N, \beta, d, m)\) does not synchronize.

VI. ACKNOWLEDGEMENT

This work was supported in part by Ministero dell’Istruzione, dell’Università e della Ricerca under PRIN Project no. 2004092944-L004.

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